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A Multiname First-Passage Model for Credit Risk*

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Abstract

In multiname extensions of the seminal Black-Cox model, dependence is typically introduced by correlating the Brownian motions driving firm values. Despite its significant intuitive appeal such a framework is simply not capable of describing market data. In this paper we propose a novel multiname framework by altering the location of systematic risk in the Black-Cox model. This is accomplished by introducing common "systematic risk" processes which govern the trend and volatility in credit qualities. We are able to calibrate several versions of the model to market quotes for CDX index tranches, including quotes from the current distressed environment.

1 Introduction

In this paper we propose a novel multiname first-passage framework for credit risk. In order to motivate this framework let us briefly re-examine the most common multiname extension of the seminal Black-Cox model. In this model a firm defaults upon first passage of its "credit quality" process,

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denoted X^i, to zero. Credit quality is defined as the log-ratio of the firm's asset value to its default threshold and is given by

$$X_t^i = x_0^i + \mu_i t + \sigma_i W_t^i \tag{1.1}$$

Here μ_i, σ_i, x_0^i are constant parameters while the W^i are correlated Brownian motion. The parameters μ_i and σ_i represent *trend* and *volatility* in credit qualities, respectively, while systematic risk appears under the guise of correlated "noise" about this trend. Despite its significant intuitive appeal the model is simply not capable of describing market data for multiname credit derivatives; in fact several authors [9, 17] have found its predictions to be quite similar to those obtained by the industry standard Gaussian copula model.

Our suspicion is that the "location" of systematic risk here is the model's fatal flaw. It is well documented that quantities such as returns, volatilities and correlations between financial assets are significantly influenced by the general state of the economy. In recognition of this fact our framework removes systematic risk from the driving Brownian motion in (1.1), and places it in trend and volatility. We allow for both stochastic trend and volatility in credit qualities, with dependence introduced by letting these quantities be governed by systematic factors common to all obligors.

In order to test the framework's flexibility we calibrate several versions to market data for CDX index tranches. We use two sets of quotes, the first consisting of pre-crisis quotes taken from November 2006 and the second consisting of more recent quotes taken from March 2008. For the 2006 data, which includes quotes for super-senior tranches, we obtain very encouraging results when the model is calibrated across both tranches and maturities simultaneously. In addition the calibrated parameters predict CDS spreads, which are not used in the calibration, that are consistent with market quotes.

For the 2008 data, which does not include super-senior or CDS quotes, we also obtain very encouraging results. As an example the model discussed in Section 4 contains ten parameters, and produces an average relative pricing error of only 3% when fitted to fifteen total spreads (each of five tranches at three different maturities). This contrasts with the failure of the industry standard model to describe this "distressed" data (see [13]). For the senior 15-30% tranches, there are simply no parameter values in the Gaussian base correlation model which are capable of producing spreads as large as those observed in early 2008. Our model is able to simultaneously price these tranches at five, seven and ten-year maturities with a maximum error of only 0.8 basis points. This is accomplished while maintaining a very reasonable fit to both equity and mezzanine tranches.

2 General Framework

In formulating structural models of credit risk, the standard approach is to begin by specifying the dynamics of a firm's asset value and default threshold, say S_t and B_t. Default is then typically defined as the first passage time of S_t to B_t (see [21] for an interesting alternative based on occupational time), equivalently the first passage time of "credit quality" $X_t = \log(S_t/B_t)$ to zero. Indeed one may take this abstract notion of credit quality as the starting point in the modeling process, and this is precisely the approach we take here.

As in the widely popular factor models, our framework makes a clear delineation between systematic and idiosyncratic sources of risk. Systematic risk is modeled via the pair of processes (M, V), which are not assumed to be independent of one another. We often refer to this pair as the "systematic factors." In order to facilitate discussion we loosely interpret these factors as being representative of the economic environment in which obligors operate, however we do not make an explicit link between these factors and any specific macroeconomic covariates.

Idiosyncratic risk is introduced via a sequence of independent Brownian motions $W^1, W^2, \ldots$, and we assume that (M, V) is independent of this sequence. Together these elements combine to drive the credit quality of obligor i as follows

$$dX_t^i = \mu_i(M_t)\,dt + \sigma_i(V_t)\,dW_t^i \qquad X_0^i = x_0^i. \tag{2.1}$$

Here $x_0^i > 0$ is a constant, while μ_i, σ_i are deterministic functions.[1] Note also that while we do not allow the initial value of credit quality to be stochastic, this added generality could be handled quite easily. Finally, we assume that obligor i defaults upon first passage of X^i to zero, that is

$$\tau_i = \inf\left\{t \geq 0 : \; X_t^i \leq 0\right\} \tag{2.2}$$

represents the default time of the i^{th} name.

A heuristic interpretation of the framework is as follows. Conditional upon the realized values of M_t, V_t, the incremental change $X_{t+h}^i - X_t^i$ is approximately Gaussian with mean $h\mu_i(M_t)$ and variance $h\sigma_i^2(V_t)$. Moreover incremental changes of distinct obligors are approximately independent over short time horizons. Thus we may think of the systematic factors as "setting the tone" for a day's operations, and once this tone has been set the fortunes of individual obligors become independent. By allowing the systematic factors to be time-varying and stochastic, we allow the economic environment in which obligors operate to evolve dynamically through time. This is not permitted in the Black-Cox model.

Our primary focus here is to investigate the model's ability to describe market data, and to gain insights into the nature of portfolio losses using realistic (i.e. calibrated) parameters. In pricing credit derivatives "today," one need only specify the information available to investors at time $t = 0$. To this end we presume that the initial value of credit quality, x_0^i, is known to investors and that the initial values of the systematic factors, (M_0, V_0), are unknown. Questions regarding the predictability of default times, existence of intensities and dynamics of credit spreads with respect to various filtrations are the subject of ongoing research, and are outside the scope of this paper.

Several existing models fit within this general framework. These include the regime-switching model proposed by Kuen et al. [14] to price credit default swaps in a single-name setting, as well as the stochastic volatility

[1] In order that X^i be well-defined we require $\mu_i(M_t)$ and $\sigma_i^2(V_t)$ to have integrable sample paths. To this end we assume that μ_i and σ_i are continuous functions, and that all sample paths of the systematic factors are càdlàg.

models proposed by Fouque et al. [7, 8] to study the effects of stochastic volatility on bond yields [7] and portfolio loss distributions [8].

In addition our framework bears some resemblance to the time change models proposed by Luciano and Schoutens [15] and Hurd [10], where asset values are modeled as independent Brownian motion subjected to a common time change. To see this, we note that we may express the law of X^i in terms of a time-changed Brownian motion with stochastic drift

$$\begin{aligned} X_t^i &= x_0^i + \int_0^t \mu_i(M_s)ds + \int_0^t \sigma_i(V_s)dW_s^i \\ &\stackrel{\mathcal{L}}{=} A_t^i + W^i(B_t^i), \end{aligned} \tag{2.3}$$

where $A_t^i = x_0^i + \int_0^t \mu_i(M_s)ds$, $B_t^i = \int_0^t \sigma_i^2(V_s)$ and $\stackrel{\mathcal{L}}{=}$ denotes equality in law. Thus default times are equivalent to first passage times of time-changed Brownian motion to stochastic barriers

$$\tau_i \stackrel{d}{=} \inf\left\{t \geq 0 : \; W^i\left(B_t^i\right) \leq -A_t^i\right\} \stackrel{d}{=} \inf\left\{t \geq 0 : \; W^i\left(B_t^i\right) \geq A_t^i\right\}, \tag{2.4}$$

where $\stackrel{d}{=}$ denotes equality in distribution.

We note that, in general, closed form expressions for default probabilities $P\left(\tau_i \leq t\right)$ are unavailable. However several numerical techniques are available for approximating first passage probabilities of (deterministically) time-changed Brownian motion to (deterministic) general barriers (see [16] for a guide to the literature). As such, provided we are able to simulate A^i and B^i, the representation (2.4), together with our conditional independence structure, allows us to approximate default probabilities and simulate portfolio losses via Monte Carlo. See Section 4.1 for a more detailed discussion.

2.1 Large Portfolio Approximation

Large portfolio approximations, pioneered by Vasicek [20], have become an important tool in credit risk. Not only can they enhance the computational efficiency of model implementation, they are also indispensable tool for gaining insights into the nature of portfolio losses.

In this section we investigate the large N asymptotics of the proportion of defaults

$$D_N(t) = \frac{1}{N}\sum_{i=1}^{N} I\left(\tau_i \le t\right). \tag{2.5}$$

Our main result (whose proof is provided in Appendix A) is Proposition 2.1 below. It is worth noting that this result relies only on the conditional independence structure of the framework, as such it remains valid in any model with conditionally independent defaults.

Proposition 2.1. *Let $\mathcal{H}_t = \sigma\left(M_s, V_s :\ 0 \le s \le t\right)$ denote the filtration generated by the systematic factors. Then for each t we have*

$$\lim_{N\to\infty}\left[D_N(t) - E\left[D_N(t)\,|\mathcal{H}_t\right]\right] = 0$$

almost surely.

Proof. See Appendix A. □

An intuitive interpretation of this result is that one is able predict the proportion of defaults in a large portfolio based solely on the information provided by the systematic factors. More formally, when it exists we call

$$D(t) = \lim_{N\to\infty} D_N(t) \tag{2.6}$$

the *asymptotic proportion of defaults*, and note that a necessary and sufficient condition for $D(t)$ to be well-defined is that the conditional expectations $E\left[D_N(t)\,|\mathcal{H}_t\right]$ converge almost surely. In this case we clearly have

$$D(t) = \lim_{N\to\infty} E\left[D_N(t)\,|\mathcal{H}_t\right],$$

so that $D(t)$ is necessarily $\mathcal{H}_t$-measurable; hence a path functional of the systematic factors. This lends formal justification to heuristic claims such as “in a large portfolio all risk is systematic” or “in a large portfolio idiosyncratic risk can be diversified away.”

Let us now suppose that credit qualities are homogeneous in the sense that $\mu_i = \mu$, $\sigma_i = \sigma$ and $x_0^i = x_0$, so that

$$dX_t^i = \mu\left(M_t\right)dt + \sigma\left(V_t\right)dW_t^i \qquad X_0^i = x_0.$$

In this situation the laws of the X^i are identical, and we clearly have that conditional default probabilities are equal across obligors

$$P\left(\tau_i \leq t \,|\mathcal{H}_t\right) = P\left(\tau_j \leq t \,|\mathcal{H}_t\right) \qquad \forall\, i, j.$$

It follows that the asymptotic proportion of defaults is given by

$$D(t) = P\left(\tau_1 \leq t \,|\mathcal{H}_t\right). \tag{2.7}$$

Hence in the homogeneous case, the asymptotic proportion of defaults is given by the conditional default probability of an arbitrary obligor.

As a second example consider a portfolio consisting of a finite number of groups, with credit qualities being homogeneous within groups. We may think of obligors within each group as belonging to the same industry, or possessing the same credit rating. Indexing the groups by $k = 1, \ldots, K$, within-group homogeneity is tantamount to the assumption that if obligor i belongs to group k, then

$$dX_t^i = \mu_k\left(M_t\right) + \sigma_k\left(V_t\right) dW_t^i \qquad X_0^i = x_0^k.$$

Now let $w_{k,N}$ denote the proportion of obligors 1 through N which belong to group k, and let P_k denote the conditional default probability of an arbitrary obligor from group k. Provided that the proportion of obligors in each group is asymptotically stable, that is $w_k := \lim_{N\to\infty} w_{k,N}$ is well-defined for each k, it is straightforward to see that

$$D(t) = \sum_{k=1}^{K} w_k P_k.$$

and we see that in such a portfolio, the asymptotic proportion of defaults is simply a weighted average of conditional default probabilities.

3 A Linear Model

In this section we investigate the simplest version of our framework, namely the homogeneous model

$$X_t^i = x_0 + Mt + \sqrt{V} W_t^i,$$

where M and V are (not necessarily independent) random variables. Recall that τ_i, the default time of obligor i, is given by the first passage time of X^i to zero.[2]

Conceptually, we might imagine that the values of the systematic factors are determined "today" ($t = 0$), yet remain unbeknownst to investors. Thus, in this model, the economic environment in which obligors operate is uncertain, but frozen in time. While we acknowledge that this lack of time dynamics conflicts with reality, this simple specification fits market data quite well. Moreover, many quantities of interest are available in closed form here, and we have found that a careful analysis of this "linear" model provides valuable insights into the nature of large portfolio losses in the more flexible dynamic model presented in Section 4.

Conditional upon the realized values of the factors, say $(M, V) = (m, v)$, the probability an arbitrary obligor defaults by time t is simply[3]

$$h(m, v, x_0, t) = \Phi\left(-\frac{x_0 + mt}{\sqrt{vt}}\right) + e^{-2x_0 m/v}\Phi\left(\frac{mt - x_0}{\sqrt{vt}}\right), \tag{3.1}$$

where Φ denotes the distribution function of a standard Gaussian variate. Recalling (2.7), we have that the asymptotic proportion of defaults is given by

$$D(t) = h(M, V, x_0, t). \tag{3.2}$$

Henceforth, we use the terms "default rate" and "conditional default probability" interchangeably.

3.1 Effect of the Systematic Factors

The effect of the factor M on the default rate is straightforward. All else being equal, as obligors drift towards the origin at a more rapid pace, the default rate increases. Indeed it is straightforward to show that $\frac{\partial h}{\partial m} < 0$ for

[2]Since $W_t^i = O\left(2t \log\log t\right)$ as $t \to \infty$ (see [11]), we have that $X_t^i \sim Mt$ as $t \to \infty$. Thus we may interpret M as the dominant long-term force driving credit qualities; when $M < 0$ eventual default for all obligors is inevitable.

[3]Note that if τ denotes the first passage time of the process $x_0 + mt + \sqrt{v}W_t$ to zero, then $h(m, v, x_0, \cdot)$ is the cumulative distribution function of τ. Here x_0, m, v are fixed parameters and W is a standard Brownian motion.

all values of the other parameters. In addition $\lim_{t\to\infty} h(m, v, x_0, t)$ is easily determined, and the proportion of obligors who *ever* default is well-defined, being given by

$$D(\infty) := \lim_{t\to\infty} D(t) = \begin{cases} 1 & M \leq 0 \, , \\ \exp\left(-\frac{2x_0 M}{V}\right) & M > 0 \, . \end{cases} \qquad (3.3)$$

It is tempting to presume that, all else being equal, an increase in volatility will produce a corresponding increase in the default rate. However this is not true, as illustrated in Figure 3.1. The most striking feature of this figure is that low volatility can (but does not necessarily) induce default rates near 100%, dispelling the notion that low volatility is unambiguously "good."

The phenomenon illustrated in Figure 3.1 is easily understood by setting $v = 0$, in which credit qualities are (identical) deterministic and linear processes $X_t^i = x_0 + mt$. In this zero-volatility situation default is either impossible or certain, according as $x_0 + mt$ is positive or negative.

When (m, x_0, t) are fixed such that $x_0 + mt > 0$, the default rate behaves "as expected," namely it is increasing in volatility, tending to 0 and 1 as v tends to 0 and ∞, respectively. When $x_0 + mt < 0$, however, the default rate behaves as in Figure 3.2. When $v = 0$ credit qualities are deterministic and default by time t is inevitable. However when v is small but non-zero, the presence of stochastic behaviour makes avoiding default possible (though not terribly likely), and default probabilities initially decrease as stochastic behaviour is introduced. Eventually this effect "wears off," and default probabilities begin to increase with volatility (and tend to 1 as v tends to ∞).

We conclude this section with a brief description of the trajectory of portfolio losses when $m < 0$ and $v \approx 0$. When $m < 0$, eventual default for each obligor is inevitable. Moreover, sending $v \to 0$ has the effect of causing all defaults to "cluster" around the critical time $t^* = -\frac{x_0}{m}$. Indeed it is straightforward to show that for $m < 0$ we have

$$\lim_{v\to 0} h(m, v, x_0, t) = \begin{cases} 0 & t < -\frac{x_0}{m} , \\ 1 & t > -\frac{x_0}{m} . \end{cases} \qquad (3.4)$$

Thus when $m < 0$ and $v \approx 0$ the trajectory of the cumulative default rate[4] will resemble a degenerate distribution function which assigns all of its mass to the point $t^* = -\frac{x_0}{m}$. In this manner the model is capable of producing scenarios where a large proportion of a portfolio defaults in a very short time interval. Despite the fact that the (asymptotic) default rate evolves continuously through time here, it is thus possible to observe dramatic "jump-type" behaviour in extreme circumstances.

3.2 Calibration to CDX Index Tranches

In order to implement the model we must specify a joint distribution for the pair (M, V). To this end we assume that both M and $\log(V)$ have a Laplace distribution. This is a flexible family of heavy-tailed, infinitely divisible and asymmetric densities, whose properties are discussed further in Appendix B.[5] We tie these marginal densities together with a Gaussian copula, leading to a seven-parameter bivariate density (three parameters for each marginal density, as well as the correlation parameter in the copula). Finally, we treat the initial value of credit quality, x_0, as a parameter to be calibrated.

In order to price tranches of synthetic CDOs we use the valuation procedure described in Appendix C. We use the large portfolio approximation and assume that recovery rates are constant at 40%. Finally, we assume the risk-free term structure is flat at 5%. Model-implied tranche spreads are approximated via Monte Carlo.

[4] When $x_0 + mt < 0$ and v is very small, computation of the term $e^{-2x_0 m/v}\Phi\left(\frac{mt-x_0}{\sqrt{vt}}\right)$ suffers from numerical instabilities. We have found the following relation quite useful in calculating conditional default probabilities in this case

$$e^{-2x_0 m/v}\Phi\left(\frac{mt-x_0}{\sqrt{vt}}\right) \sim \frac{\sqrt{vt}}{x_0 - mt}\phi\left(\frac{x_0+mt}{\sqrt{vt}}\right) \qquad \text{as } v \to 0\ .$$

This can be checked using the well-known relation $1 - \Phi(x) \sim \frac{\phi(x)}{x}$ as $x \to \infty$.

[5] In our experimentation we have found that heavy-tailed marginal distributions are important for producing non-negligible spreads for senior tranches, and that asymmetries in the marginal densities are critical for accurate simultaneous pricing of junior and senior tranches.

The calibration procedure used is as follows. With θ denoting the vector of model parameters (of which there are eight in total) we minimize the mean relative error

$$d(\theta) = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{\left| s_{i,j}^{mkt} - s_{i,j}^{mod}(\theta) \right|}{s_{i,j}^{mod}}$$

where i and j index tranche and maturity, respectively, and with s^{mkt} and s^{mod} denoting market and model-implied spreads, respectively. The minimization is carried out using the "simulated annealing" algorithm described by Corana et al. [5]. As the surface $d(\theta)$ tends to be highly irregular, we have found this stochastic optimization algorithm vastly superior (in terms of locating minima) to the "greedy" Nelder-Mead, however this comes at the cost of significantly more function evaluations.

Calibration to 2006 data. Table 3.1 presents our preliminary calibration results. The data was obtained from [6] and consists of market quotes for the CDX index (Investment Grade Series 7) and CDX index tranches at each of three maturities, as of November 1, 2006. In [6] the authors calibrate a reduced-form model consisting of 32 parameters, and obtain a more or less perfect fit. Also note that we do not include the index spread in our calibration; parameters were only chosen to reproduce multiname instruments as faithfully as possible.[6]

The results presented here are quite encouraging. We obtain a reasonable fit across both maturities and tranches, though relative pricing errors do tend to increase with seniority. Of particular note is the fact that index spreads, which were not included in the calibration, are priced quite accurately.

Large Portfolio Losses. In order to gain insights into the nature of large portfolio losses here, Figure 3.3 presents the contours of the conditional bivariate density of the systematic factors, conditioned upon the super-senior

[6]In a homogeneous and equally-weighted portfolio the fair spread on the index will be the same the fair spread on any of the individual single-name CDS. Thus, from a pricing perspective, the index is effectively a single-name instrument.

tranche (30-100%) experiencing losses (with our assumption of a constant recovery rate of 40%, this is equivalent to the default rate exceeding 50%).

As might be expected, we see that the conditional distribution assigns virtually all of its mass to the region where M drives credit qualities rapidly towards default. The most striking feature of this distribution, however, is that these scenarios also tend to be characterized by "negligible" values of V, that is large portfolio losses tend to be characterized by periods of abnormally *low* volatility in credit qualities. These "low volatility" crashes were explained mathematically in Section 3.1; here we give a more "economic" interpretation.

To begin note that when $V \approx 0$ we have that credit qualities evolve as

$$X_t^i = x_0 + Mt + \sqrt{V} W_t^i \approx x_0 + Mt. \tag{3.5}$$

Thus during the most severe downturns, the influence of the idiosyncratic component is negligible; there is very little that an individual obligor can do to "distinguish herself from the pack." Alternatively we may interpret (3.5) as saying that in a downturn, all risk is systematic and diversification is impossible; the behaviour of an individual obligor is virtually indistinguishable from the average.

In Section 3.1 we noted that, when $M < 0$ and $V \approx 0$, the trajectory of portfolio losses will resemble a degenerate distribution function. Figure 3.4 illustrates a representative trajectory for the default rate in one of these scenarios.

Calibration to 2008 ("distressed") data. Recent times have seen significant turmoil in markets for credit derivatives. Krekel [13] provides a set of CDX index (Investment Grade Series 9) tranche quotes from March 10, 2008, stating

> In February and March 2008 it was temporarily not possible to calibrate the standard Gaussian base correlation model to the complete set of CDX and iTraxx tranche quotes. The reason is that the Gaussian base correlation model was not able to gener-

ate enough probability for high portfolio losses, while preserving the calibration to mezzanine and equity tranches.

Table 3.2 presents the results of calibrating the linear model to this "distressed" data. We note that neither super-senior tranche nor index spreads were available. The results are again encouraging, with the overall relative pricing error at 5.3%. The model is able to capture the senior spreads reasonably well, while at the same time maintaining an acceptable fit to the equity and mezzanine tranches.

Parameter Comparison. As might be expected the initial value of credit quality is much lower for the distressed data, which is consistent with the perception that obligors are currently in a much more precarious position financially. Figure 3.5 compares the calibrated marginal densities for each of M and $\log(V)$. As might be expected the distressed density for M is skewed more heavily towards negative values, however the most striking difference lies in the calibrated densities for V. The distressed density is *much* more heavily concentrated around small values, giving the appearance that idiosyncratic risk has been "priced out" during the current turmoil. Indeed, it is tempting to conclude that the primary "role" of the factor V in our calibration exercise is to downgrade the idiosyncratic component of credit quality in periods of extreme economic turbulence.

4 A Dynamic Model

In the linear model of Section 3 the economic environment in which obligors operate is ostensibly determined "today," and is not permitted to evolve over time. In this section we introduce time dynamics to this environment, investigating the homogeneous model

$$dX_t^i = M_t dt + \sqrt{V_t} dW_t^i \qquad X_0^i = x_0,$$

where M_t and V_t are (not necessarily independent) processes with integrable sample paths.

The introduction of dynamic risk factors allows for much richer dynamics in the default rate, as illustrated in Figure 4.1. This figure presents simulated trajectories (using parameters calibrated to the 2006 data) for the systematic factors, as well as the corresponding trajectory for the default rate, from the model calibrated in Section 4.2. In this model the factors are correlated mean-reverting diffusion processes. We observe two distinct "clusters" of defaults in Figure 4.1, a phenomenon which is not possible in the linear model.[7]

4.1 Conditional Default Probabilities

The primary challenge in implementing the dynamic model is the simulation of conditional default probabilities. For deterministic functions $f : [0, T] \to \mathbb{R}$ and $g : [0, T] \to (0, \infty)$, we define

$$\Psi(f, g, t) = P\left(\min_{0 \le s \le t} \{f(s) + W(g(s))\} \le 0\right), \tag{4.1}$$

where W is a standard Brownian motion. We assume $f(0) > 0$ and note that Ψ returns the probability that a Brownian motion, subjected to the time-change g, strikes the upper barrier f at any point over the interval $[0, t]$. In addition, for fixed f and g, $\Psi(f, g, \cdot)$ defines the distribution function of the first passage time of the time-changed Brownian motion to the barrier.

In light of (2.4), we may express conditional default probabilities, and hence the asymptotic proportion of defaults, in our dynamic model via

$$P(\tau_i \le t \,|\mathcal{H}_t) = \Psi(A, B, t), \tag{4.2}$$

where $A_t = x_0 + \int_0^t M_s ds$ and $B_t = \int_0^t V_s ds$. Thus a general program for simulating default trajectories would be to first simulate realizations of A and B, say a and b, and then compute $\Psi(a, b, t)$ for various values of t. Unfortunately Ψ is rarely available in closed form, however its approximation is a very well developed problem in the literature, where a variety of methods

[7] The reason for this is that, in the linear model, trajectories for the default rate correspond to distribution functions whose densities are unimodal.

have been proposed (see [16] for a more detailed discussion and guide to the literature).

When f and g are continuously differentiable (as they will be in our ultimate application), with $g' > 0$, it is known [18] that $\Psi(f, g, \cdot)$ solves the following Volterra equation of the first kind

$$\Phi\left(-\frac{f(t)}{\sqrt{g(t)}}\right) = \int_0^t K(s,t)\,\Psi(f,g,ds)\ , \tag{4.3}$$

where

$$K(s,t) = \Phi\left(-\frac{f(t)-f(s)}{\sqrt{g(t)-g(s)}}\right)$$

for $s < t$ and Φ denotes the distribution function of a standard Gaussian variate. Noting that $K(s,t) \to 1/2$ as $s \nearrow t$, we see that the kernel here is bounded, and a rapid approximation to Ψ at a given sequence of points $t_1 < t_2 < \ldots t_n$ may be obtained by making the following observation

$$\Phi\left(-\frac{f(t_i)}{\sqrt{g(t_i)}}\right) \approx \sum_{j=1}^{i} K(t_j,t_i)\left[\Psi(f,g,t_j) - \Psi(f,g,t_{j-1})\right]$$

where we set $K(t,t) = 1/2$ and $t_0 = 0$, noting that $\Psi(f,g,0) = 0$ (we have assumed $f(0) > 0$). The following system of equations is rapidly solved numerically

$$\Phi\left(-\frac{f(t_i)}{\sqrt{g(t_i)}}\right) = \sum_{j=1}^{i} K(t_j,t_i)\left[d_j - d_{j-1}\right]\ , \tag{4.4}$$

leading to the approximations $d_i \approx \Psi(f,g,t_i)$.

An alternative to the integral equation approach is to approximate the barriers with piecewise linear functions, in which case Ψ is available semi-analytically and may be approximated with recursive quadrature or Monte Carlo simulation. While approximations based on integral equations tend to be *much* more rapid, the advantage of the piecewise linear approach is that it is easily extended to the case where f and g possess discontinuities (of the first kind).

4.2 Calibration to CDX Index Tranches

In order to implement the model we must specify the dynamics of the systematic factors. To this end we model M_t and V_t are mean-reverting diffusion processes

$$\begin{aligned} dM_t &= \theta_M \left(\mu_M - M_t\right) dt + \sigma_M \left(M_t\right) dZ_t^1 \ , \\ dV_t &= \theta_V \left(\mu_V - V_t\right) dt + \sigma_V \left(V_t\right) dZ_t^2 \ . \end{aligned}$$

Here Z^1, Z^2 are correlated standard Brownian motion, independent of the idiosyncratic sequence $W^1, W^2, \ldots$. Owing to the success of the Laplace and log-Laplace distributions in the previous section, we use the results of Bibby et al. [3] to choose the functions σ_M, σ_V in such a way as to ensure M and V are ergodic, with Laplace and log-Laplace invariant densities, respectively. Details of these constructions are contained in Appendix B, and explicit expressions for σ_M and σ_V are given by (B.5) and (B.6), respectively.

There are ten parameters in this model. These include the initial value of credit quality x_0 as well as the correlation ρ between Z^1 and Z^2. Each systematic factor also has four "marginal" parameters. These include the mean reversion rates θ_M and θ_V, as well as the three parameters governing each invariant density. These parameters influence both the stationary distributions, on which their impact is straightforward, as well as the diffusion coefficients σ_M and σ_V, on which their impact is more subtle.

In order to simulate trajectories of portfolio losses we use the following program

- Simulate trajectories for the systematic factors. We use a simple Euler scheme with a time-step of $dt = 10^{-2}$, and in an effort to draw (M_0, V_0) from its stationary distribution[8] we "begin" the simulation five years prior to time $t = 0$. That is we set $(M_{-5}, V_{-5}) = (\mu_M, \mu_V)$ and use our Euler scheme to simulate (M_0, V_0).

[8] Owing to the correlation between the driving Brownian motion, this stationary distribution is non-trivial, despite the fact that each marginal stationary distribution is available.

- Use quadrature to reconstruct the integrals A_t and B_t. To this end we use the simple trapezoidal rule.

- Approximate Ψ by solving the system (4.4). In the name of computational efficiency we use $t_i = i\Delta$ with $\Delta = 0.25$, noting that our mesh for approximating Ψ is coarser than our mesh for simulating the factor paths.

Calibration to 2006 data. Table 4.1 presents our calibration results for the "non-distressed" 2006 data. We obtain a superior overall fit than with the linear model, with the average relative pricing error decreasing from 16.45% to 15.14%. Despite the fact that our overall fit improves, however, we obtain a slightly poorer fit to the super-senior tranches.

In the linear model the default rate was available in closed form, which permitted a careful analysis of large portfolio losses. Owing to the analytic intractability of the default rate in the dynamic model, however, a similar analysis is not possible here. Nonetheless one may gain insights into the nature of large portfolio losses in the dynamic model by simulating a large number of trajectories for the systematic factors and inspecting those which produce large losses.

In carrying out such a program, one finds that large losses tend to be characterized by scenarios where M_t spends a prolonged period of time below zero, while V_t simultaneously spends a prolonged period of time near zero. A representative "large portfolio loss" scenario is illustrated in Figure 4.2. It appears that, as in the linear model, the primary role played by the factor V_t here is to downgrade the idiosyncratic component of credit quality during severe market downturns. Unlike in the linear model, however, it is possible for the economic environment to "recover."

Calibration to 2008 data. Table 4.2 presents the results of two different calibration exercises for the 2008 data. In the first exercise, the results of which are labelled "Static" in the table, we calibrate all ten diffusion parameters to the data. We obtain a superior fit when compared to the linear model, with the mean relative pricing error decreasing from 5.3% to

3.36%. In addition the senior tranches are priced to a *total* error of only 1.2 basis points, with the maximum pricing error of 0.8 basis points occurring with the seven-year tranche.

Our second calibration exercise is motivated by the fact that in our dynamic model, changes in credit spreads should be driven by changes in the values of the systematic factors themselves, and not changes in the "fixed" parameters governing their temporal evolution. That is, parameters calibrated to one set of data should be capable of describing spreads observed at a different point in time.

To this end, we suppose that the model is "correct," in that it fully captures all relevant features of the credit environment. We further suppose that the diffusion parameters calibrated to the 2006 data are in fact the true parameters governing the evolution of the systematic factors. Finally, we suppose that the initial values (x_0, M_0, V_0) are known to investors, where $t = 0$ corresponds to the valuation date of March 10, 2008. This effectively yields a three-parameter model which we calibrate to the 2008 data, obtaining the results labelled "Dynamic" in Table 4.2. While the fit obtained here is admittedly less than perfect, we feel it is quite encouraging in light of the fact that we are only fitting three parameters to fifteen data points. Indeed, we feel these results provide a promising step forward in modeling the joint dynamics of credit spreads.

5 Conclusion

This paper proposes a novel multiname first-passage framework for credit risk. In analogy with the seminal Black-Cox model, default occurs upon first passage of an obligor's "credit quality" process to zero. Credit qualities are modeled as continuous processes, with dependence introduced via systematic risk processes governing local trend (drift) and volatility. The result is a dynamic credit environment which permits flexible time dynamics in the default rate. This contrasts the Black-Cox model, where dependence is typically introduced via "correlated Brownian noise" about a deterministic trend.

We calibrate several versions of the model to market data for CDX index tranches, using data from both 2006 (a period of relative calm) and 2008 (a period of virtually unprecedented turmoil). We obtain encouraging results for both sets of data, calibrating across both tranches and maturities simultaneously. This provides an example of a purely continuous framework which is able to successfully reproduce the entire (market-implied) term structure of (expected) portfolio losses.

Using calibrated parameters, we find that large portfolio losses require a combination of sustained downward trend (negative drift) and low volatility. While the former phenomenon is to be expected, the latter phenomenon may appear surprising; it is tempting to presume that low volatility is unambiguously "good." However a brief analysis reveals that periods of low volatility can also be interpreted as periods where the idiosyncratic component of credit quality is negligible. Thus in the presence of a sustained downward trend, low volatility can have disastrous consequences.

We feel that the results of our study are a promising step forward for modeling the joint dynamics of credit spreads and default rates. As discussed in the paper, it appears as though the primary role played by our volatility factor is to downgrade idiosyncratic influence in periods of extreme economic turbulence. In addition, our framework provides a natural dynamic extension of the widely popular factor models. It has been noted [2] that incorporating "random factor loadings," or "stochastic correlation," dramatically enhances the ability of such models to describe observed correlation skews. We feel that our results highlight the importance of incorporating analogous concepts when building dynamic multiname credit models.

APPENDICES

A Proof of Proposition 2.1

Suppose that $X_1, X_2, \ldots$ is a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, P)$, with finite second moments. We say the sequence is orthogonal if $E[X_i X_j] = E[X_i] E[X_j]$ for all $i \neq j$. In order to prove Proposition 2.1, we require the following

Theorem A.1 (Stout [19]). *Suppose that $X_1, X_2, \ldots$ is an orthogonal sequence such that*

$$\sum_{n=1}^{\infty} \log^2(n) E\left[X_n^2\right] < \infty \ .$$

Then $S_n = \sum_{i=1}^{n} X_i$ converges almost surely.

We say that our sequence is conditionally orthogonal, given $\mathcal{G} \subset \mathcal{F}$, if $E[X_i X_j | \mathcal{G}] = E[X_i | \mathcal{G}] E[X_j | \mathcal{G}]$ for all $i \neq j$. When the X_i are conditionally orthogonal Bernoulli variables, we may interpret $\frac{1}{n} \sum_{i=1}^{n} X_i$ as the proportion of successes in n trials. In general this sequence may fail to converge.[9] Theorem A.2 demonstrates that a necessary and sufficient condition for convergence is the convergence of $\frac{1}{n} \sum_{i=1}^{n} E[X_i | \mathcal{G}]$. Moreover, when it exists the limiting proportion of successes is necessarily $\mathcal{G}$-measurable.

Theorem A.2. *Suppose that $X_1, X_2, \ldots$ is a sequence of conditionally orthogonal Bernoulli variables. Then with probability one we have*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left(X_i - E[X_i | \mathcal{G}]\right) = 0 \ . \qquad \text{(A.1)}$$

[9] To see this first consider the deterministic sequence

$$0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{2}{5}, \frac{3}{6}, \frac{3}{7}, \ldots, \frac{3}{12}, \frac{4}{13}, \ldots, \frac{9}{18}, \ldots \ ,$$

which oscillates between 1/2 and 1/4. Next, set $x_1 = 0$ and $x_i = d_i - d_{i-1}$ for $i \geq 2$, where d_i is the numerator of the i^{th} term in the above sequence. Thus $\frac{1}{n} \sum_{i=1}^{n} x_i$ fails to converge. Finally, set $X_i = U x_i + (1 - U) y_i$, where U is a Bernoulli variable and $\{y_i\}_{i=1}^{\infty}$ is any sequence of zeros and ones.

Proof. To begin we denote $\tilde{X}_i = E[X_i | \mathcal{G}]$. A simple application of the tower property, combined with $\mathcal{G}$-measurability of $\tilde{X}_j$ and the fact that the X_i are conditionally uncorrelated yields, for any $i \neq j$

$$E\left[X_i \tilde{X}_j\right] = E\left[\tilde{X}_i \tilde{X}_j\right] = E[X_i X_j] \ .$$

Next we define $Y_i = (X_i - \tilde{X}_i)/i$, and note that the Y_i form an orthogonal sequence with second moments bounded by $E\left[Y_i^2\right] \leq 1/i^2$. Using the integral and comparison tests for series it follows that

$$\sum_{i=1}^{\infty} \log^2(i) E\left[Y_i^2\right] < \infty \ .$$

Invoking Theorem A.1 we see that $S_n = \sum_{i=1}^{n} Y_i$ converges almost surely to some random variable S. Applying Kronecker's Lemma we obtain the desired result. □

Proposition 2.1 now follows easily by setting $X_i = I(\tau_i \leq t)$ and $\mathcal{G} = \mathcal{H}_t$. We wish to stress the fact that Theorem A.2 may be applied to *any* model in which default times are conditionally independent. This includes the widely popular factor models, as well as many intensity-based models such as those based on Cox processes.

B The Laplace and log-Laplace Distributions

The Laplace Distribution. A random variable X is said to have a *Laplace distribution* if it has a probability density function of the form

$$f(x) = \begin{cases} [\beta_1 + \beta_2]^{-1} e^{(x-\alpha)/\beta_2} & -\infty < x \leq \alpha \ , \\ \\ [\beta_1 + \beta_2]^{-1} e^{(\alpha - x)/\beta_1} & \alpha \leq x < \infty \ . \end{cases} \tag{B.1}$$

This is a three-parameter family with a location parameter $\alpha \in \mathbb{R}$ and scale-type parameters $\beta_1, \beta_2 > 0$. The Laplace density obtains as the density of the random variable $X = \alpha + \beta_1 Z_1 - \beta_2 Z_2$, where Z_1 and Z_2 are independent unit-mean exponential variates. Infinite divisibility is immediate from this representation, as is the fact that the mean of (B.1) is $\alpha + \beta_1 - \beta_2$. Several studies have found the Laplace to be quite capable of describing financial quantities such as log-changes in interest rates [12]. A comprehensive treatment of this and related distributions is available in Kotz et al. [12].

The log-Laplace Distribution. If X has Laplace density (B.1), the distribution of $Y = e^X$ is called the log-Laplace and is easily seen to have density

$$g(y) = \begin{cases} [\beta_1 + \beta_2]^{-1} \, e^{-\alpha/\beta_2} y^{(1-\beta_2)/\beta_2} & 0 < y \le e^{\alpha} \; , \\ \\ [\beta_1 + \beta_2]^{-1} \, e^{\alpha/\beta_1} y^{-(1+\beta_1)/\beta_1} & y \ge e^{\alpha} \; . \end{cases} \tag{B.2}$$

Interesting features of the log-Laplace distribution are that it has power tails at both zero and infinity, in particular it has heavier right-hand tails than the gamma distribution. In addition this family is clearly invariant to scaling and exponentiation. Existence of the log-Laplace moments is governed solely by the parameter β_1. Indeed it is straightforward to show that for $k > 0$, $E\left[Y^k\right] < \infty$ if and only if $k < \beta_1^{-1}$. In particular the log-Laplace mean and variance exist provided $\beta_1 < 1$ and $\beta_1 < 1/2$, respectively. Finally, we note that (B.2) is bounded provided $\beta_2 \le 1$.

Constructing Ergodic Laplace Diffusion Processes. Our goal in this section is to construct an ergodic diffusion process X_t possessing (B.1) as its invariant density. Our main tool will be the results of Bibby et al. [3], which we outline here. Suppose that f is a probability density which is continuous, bounded, strictly positive on the interval (l, u) and zero outside (l, u). Here $-\infty \le l < u \le \infty$ and we require f to have a finite mean μ. For a given $\theta > 0$ define the function

$$\sigma^2(x) := \frac{2\theta \int_l^x (\mu - z) \, f(z) dz}{f(x)} \; . \tag{B.3}$$

The domain of σ is taken as (l, u). Bibby et al. [3] show that the stochastic differential equation

$$dX_t = \theta \left(\mu - X_t\right) dt + \sigma\left(X_t\right) dW_t \tag{B.4}$$

has a unique weak solution with invariant density f. Moreover, provided f has finite second moment, the correlation between X_t and X_s is equal to $e^{-\theta|t-s|}$. Clearly, if $X_0 \sim f$ then X_t is stationary as well.

The Laplace density (B.1) satisfies the conditions required for this result, and it is straightforward to verify that in this case $\mu = \alpha + \beta_1 - \beta_2$ and

$$\sigma^2(x) = \begin{cases} 2\theta\beta_2 \left[\beta_1 + \alpha - x\right] & x \leq \alpha \,, \\ 2\theta\beta_1 \left[\beta_2 + x - \alpha\right] & x > \alpha \,. \end{cases} \tag{B.5}$$

When simulating trajectories for such a process using an Euler or Milstein scheme, we have found it useful[10] to re-express (B.5) as follows

$$\sigma^2(x) = \frac{2\theta}{\sigma^2 - \kappa^2} \left[1 + \sigma \left|x - \alpha\right| + \kappa \left(x - \alpha\right)\right] \,,$$

where

$$\sigma = \frac{1}{2} \frac{\beta_1 + \beta_2}{\beta_1\beta_2} \,, \qquad \kappa = \frac{1}{2} \frac{\beta_1 - \beta_2}{\beta_1\beta_2} \,.$$

Constructing Ergodic log-Laplace Diffusion Processes. In the case of the Laplace distribution one can construct an ergodic diffusion with (B.1) as its invariant density for all values of the parameters. The same can not be said of the log-Laplace density (B.2). In order that a log-Laplace density have finite mean we require $\beta_1 < 1$, and in order that such a density be bounded we require $\beta_2 < 1$.

Using the results from Bibby et al. [3] discussed in the previous section, and with f given by (B.2) with $\max(\beta_1, \beta_2) < 1$, we have that for $\theta > 0$ the (unique weak) solution to

$$dX_t = \theta\left(\mu - X_t\right) dt + \sigma\left(X_t\right) dW_t$$

has f as its invariant density, where $\mu = e^\alpha / \left(1 - \beta_1\right)\left(1 + \beta_2\right)$ and

$$\sigma^2(x) = \begin{cases} 2\theta\beta_2 x \left[\mu - \frac{x}{1+\beta_2}\right] & 0 \leq x \leq e^\alpha \,, \\ \\ 2\theta\beta_1 x \left[\frac{x}{1-\beta_1} - \mu\right] & x \geq e^\alpha \,. \end{cases} \tag{B.6}$$

[10] The reason is that, using (B.5), one needs to identify at each time step those values of X_t which are above or below α. Though this operation is quite simple, the time required can significantly slow down the simulation process.

This can be expressed in the more compact form

$$\sigma^2(x) = 2\theta x \left[\sigma\left(x - e^{\alpha}\right) + \kappa\left|x - e^{\alpha}\right| + \mu\beta_1\beta_2\right] ,$$

where

$$\sigma = \frac{\beta_1 + 2\beta_1\beta_2 - \beta_2}{2\left(1-\beta_1\right)\left(1+\beta_2\right)} , \qquad \kappa = \frac{\beta_1 + \beta_2}{2\left(1-\beta_1\right)\left(1+\beta_2\right)} .$$

C Valuation of Synthetic CDOs

In this appendix we briefly outline the valuation procedure used in the paper. The reader who wishes a more detailed discussion on valuation of CDO tranches is referred to [9].

For a synthetic CDO tranche defined defined by an attachment point K_A and detachment point K_D, we define the present value of the protection leg as

$$PVD(\mathbf{L}) = \sum_{i=1}^{n} d(t_i)\left[f\left(L(t_i), K_A, K_D\right) - f\left(L(t_{i-1}), K_A, K_D\right)\right] ,$$

where t_i are the coupon dates, $d(t)$ is the discount factor applicable risk-free cash flows received at t, $L(t)$ is the cumulative percentage loss on the underlying portfolio, $\mathbf{L} = (L(t_0), L(t_1), \ldots L(t_n))$ is the trajectory of portfolio losses and f is the tranche loss (per dollar of underlying principal)

$$f(L, K_A, K_D) = \min\left(\max\left(0, L - K_A\right), K_D - K_A\right).$$

Note that this formula assumes that all losses incurred over the interval $(t_{i-1}, t_i]$ are reimbursed at t_i. In addition, we define the present value of the protection leg as

$$PVP(\mathbf{L}) = \sum_{i=1}^{n}(t_i - t_{i-1})d(t_i)\frac{g(L(t_i), K_A, K_D)) + g(L(t_{i-1}), K_A, K_D)}{2} ,$$

where g is defined as the outstanding tranche principal (per dollar of underlying principal)

$$g(L, K_A, K_D) = (K_D - K_A) - f(L, K_A, K_D).$$

Note that this formula assumes that the coupon due at time t_i is assessed on the simple average outstanding principal over the payment interval, and ignores accruals.

The fair spread on a non-equity tranche ($K_A > 0$) is defined as

$$s = \frac{E\left[PVD\left(\mathbf{L}\right)\right]}{E\left[PVP\left(\mathbf{L}\right)\right]},$$

while for the equity tranche ($K_A = 0$) it is defined as

$$s = \frac{E\left[PVD\left(\mathbf{L}\right)\right] - .05E\left[PVP\left(\mathbf{L}\right)\right]}{K_D}.$$

In our applications we assume the risk-free term structure is flat, and that recovery rates are deterministic and constant across obligors. With these simplifying assumptions and the large portfolio approximation we have that $d(t_i) = e^{-rt_i}$ and $L(t) = (1-R)D(t)$, where $D(t)$ is the asymptotic proportion of defaults. In addition, we assume that coupon dates are equally-spaced at three month intervals.

In order to approximate fair spread via Monte Carlo, one may simulate J independent trajectories of portfolio losses, say $\mathbf{L}_j$, compute the associated values of the two payment legs, say $PVD\left(\mathbf{L}_j\right)$ and $PVP\left(\mathbf{L}_j\right)$, and approximate the desired expectations via their empirical counterparts. For example

$$\hat{s} = \frac{\frac{1}{J}\sum_{j=1}^{J} PVD\left(\mathbf{L}_j\right)}{\frac{1}{J}\sum_{j=1}^{J} PVP\left(\mathbf{L}_j\right)}$$

provides an estimate of the fair tranche spread for a non-equity tranche, with an analogous formula holding for the equity tranche.

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Figure 3.1: The conditional default probability surface $h(m, v, x_0, t)$ for fixed values $x_0 = 1.8$ and $t = 5$.

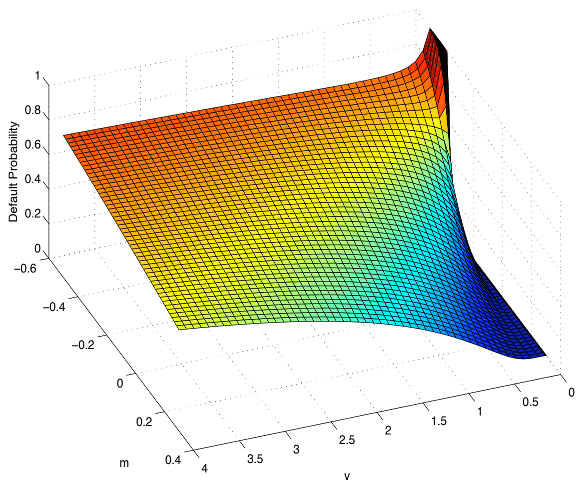

Figure 3.2: Effect of Volatility on Default Probabilities. The function $h(m, \cdot, x_0, t)$ for fixed values $m = -0.4$, $x_0 = 0.6$ and $t = 3$. Note that $x_0 + mt < 0$.

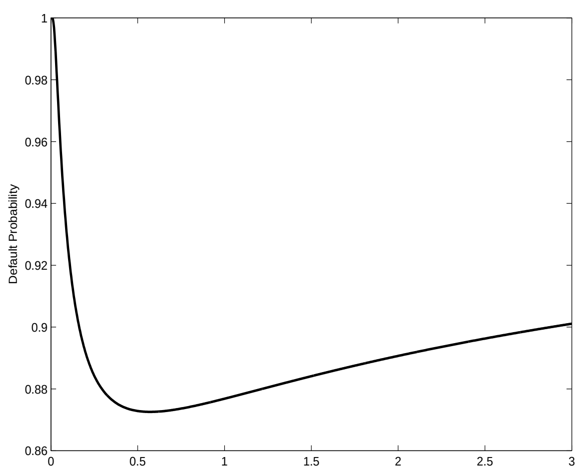

Table 3.1: Calibration to CDX Data - Linear Model. Equity spreads expressed in percentage points and quoted as upfront fees with 500 basis point running premium. All other spreads expressed in basis points and quoted as running premia.

	5Y						
	0-3%	3-7%	7-10%	10-15%	15-30%	30-100%	Index
Market	24.38	90	19	7	3.5	1.73	35
Model	24.43	90.2	17.5	7	2.5	0.38	34.8
	7Y						
	0-3%	3-7%	7-10%	10-15%	15-30%	30-100%	Index
Market	40.44	209	46	20	5.75	3.12	45
Model	40.61	250.5	45	20	9.3	2	47.3
	10Y						
	0-3%	3-7%	7-10%	10-15%	15-30%	30-100%	Index
Market	51.25	471	112	53	14	4	57
Model	49.1	471.1	112	44	19.8	4	57.5

Marginal Parameters			
	α	β_1	β_2
M	.0835	.0514	.0706
V	-1.4958	.2809	.6399

Other Parameters	
x_0	ρ
1.8371	.8908

Figure 3.3: This figure illustrates the contours of the unconditional bivariate density of (M, V), as well as the contours of the conditional bivariate density, conditioned upon the five-year super-senior tranche (30-100%) experiencing losses.

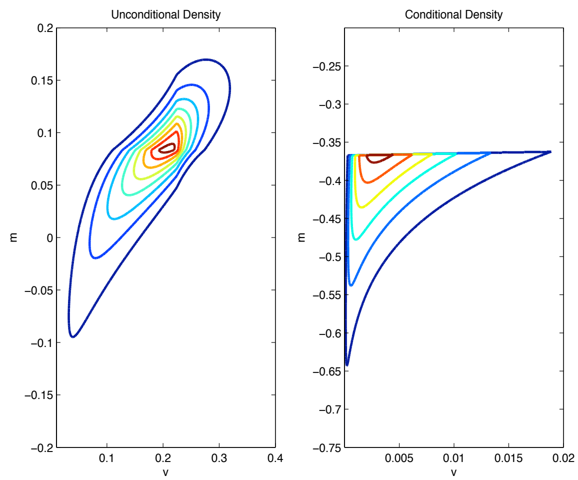

Figure 3.4: A representative trajectory for portfolio losses in the event that the five-year default rate exceeds 50%. The values used are $M = -0.375$ and $V = 0.003$, and the critical time is $t^* = -4.899$.

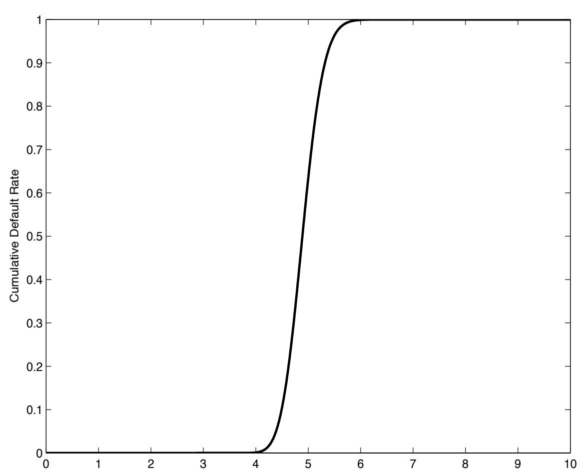

Table 3.2: Calibration to Distressed CDX Data - Linear Model. Equity spreads expressed in percentage points and quoted as upfront fees with 500 basis point running premium. All other spreads expressed in basis points and quoted as running premia.

	5Y				
	0-3%	3-7%	7-10%	10-15%	15-30%
Market	67.38	727	403	204	115
Model	65.90	733	355	219	100
	7Y				
	0-3%	3-7%	7-10%	10-15%	15-30%
Market	70.5	780	440	248	128.5
Model	70.79	859	417	265	128.1
	10Y				
	0-3%	3-7%	7-10%	10-15%	15-30%
Market	73.5	895.5	509	282	139.5
Model	71.76	894.7	430	277	141.2

Marginal Parameters			
	α	β_1	β_2
M	.0831	.01	.0534
V	-3.2536	.0271	.1455

Other Parameters	
x_0	ρ
0.5865	.8217

Figure 3.5: Calibrated Marginal Densities: The upper and lower panels compare the calibrated densities of M and $\log(V)$, respectively.

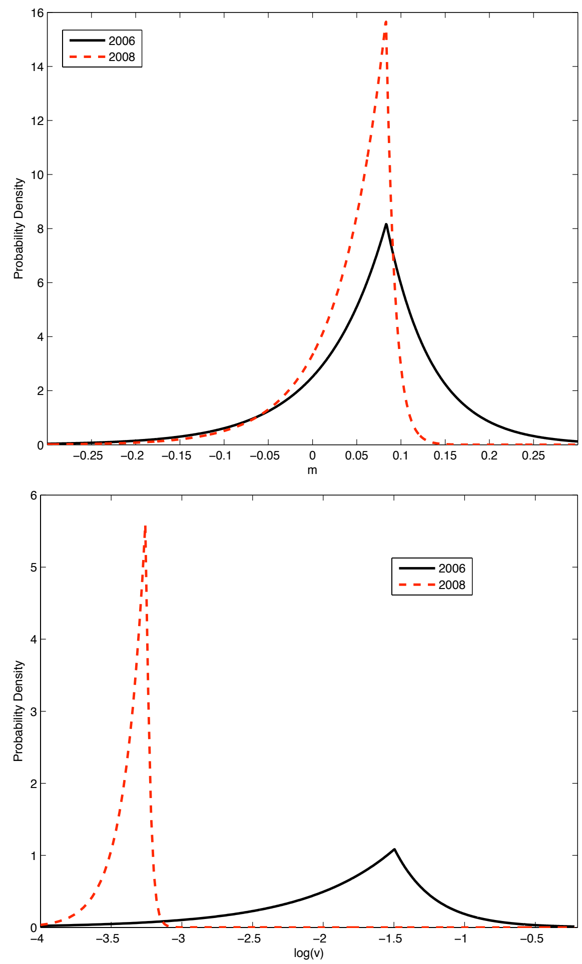

Figure 4.1: Illustrative Trajectories - Dynamic Model. This figure presents trajectories of the systematic factors and cumulative default rate, from the model calibrated in Section 4.2. Parameters are those calibrated to the 2006 data. The factors are modeled as correlated mean-reverting diffusions, with long-run mean levels illustrated with dashed lines. Note that the default rate peaks when M_t and V_t simultaneously spend prolonged periods below zero and near zero, respectively.

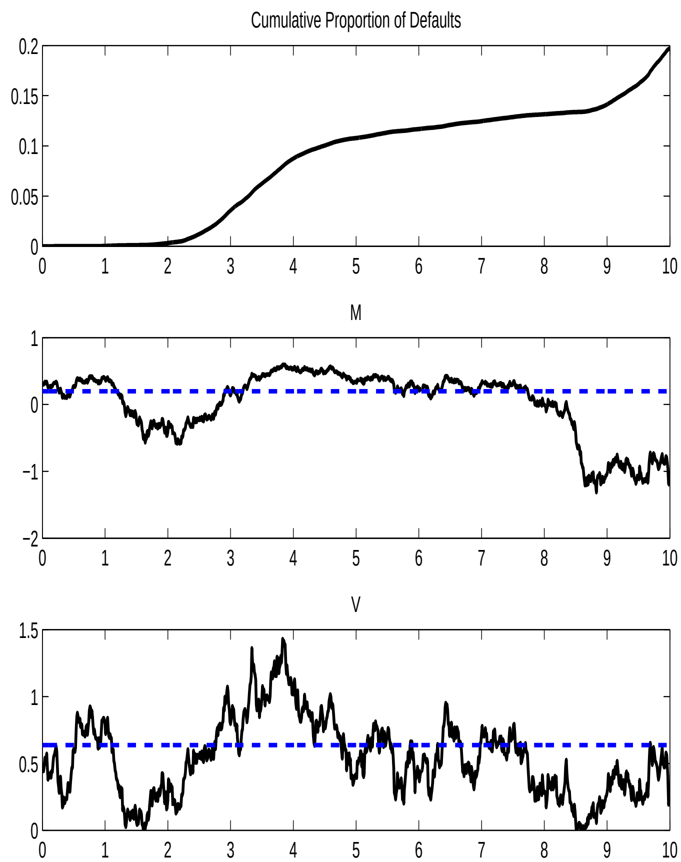

Table 4.1: Calibration to 2006 CDX Data - Diffusion Model. Equity spreads expressed in percentage points and quoted as upfront fees with 500 basis point running premium. All other spreads expressed in basis points and quoted as running premia.

	5Y						
	0-3%	3-7%	7-10%	10-15%	15-30%	30-100%	CDS
Market	24.38	90	19	7	3.5	1.73	35
Model	22.30	89.4	19.1	8	3.5	0.36	33.6
	7Y						
	0-3%	3-7%	7-10%	10-15%	15-30%	30-100%	CDS
Market	40.44	209	46	20	5.75	3.12	45
Model	40.90	235.7	46.5	18.9	6.21	1.39	46
	10Y						
	0-3%	3-7%	7-10%	10-15%	15-30%	30-100%	CDS
Market	51.25	471	112	53	14	4	57
Model	51.04	471	110.5	42	14	1.49	55.4

Marginal Parameters

	θ	α	β_1	β_2
M	.5657	.0565	.0322	.0613
V	.1252	-2.6641	.3639	.7468

Other Parameters

x_0	ρ
1.1155	.9810

Figure 4.2: This figure illustrates the behaviour of credit qualities during a "severe downturn" in our calibrated dynamic model. We note that the ten-year default rate exceeds 40% in this scenario. The top panel superimposes trajectories of five credit qualities (corresponding to distinct obligors) on the general trend in credit qualities ($\int_0^t M_s ds$). The bottom panel illustrates the corresponding trajectory of V_t. We note that during the "downturn" V_t is very small and there is very little an individual obligor can do to "distinguish herself from the pack." The idiosyncratic component "reappears" once the worst is over.

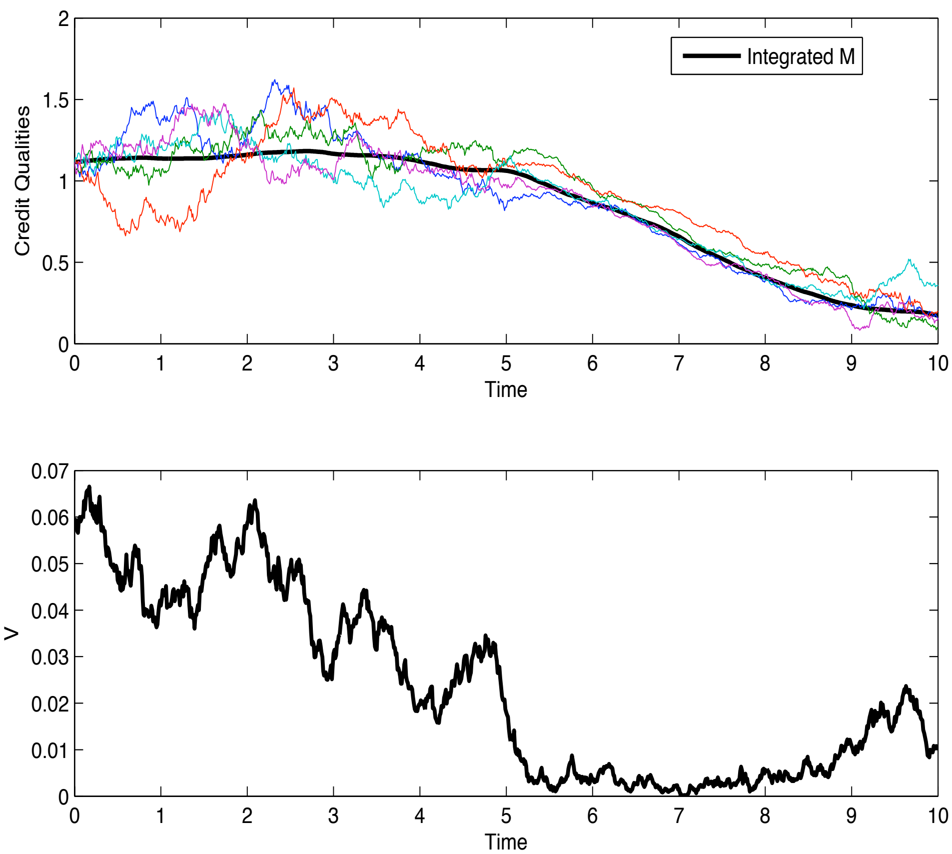

Table 4.2: Calibration to 2008 CDX Data - Diffusion Model. Equity spreads expressed in percentage points and quoted as upfront fees with 500 basis point running premium. All other spreads expressed in basis points and quoted as running premia.

	5Y				
	0-3%	3-7%	7-10%	10-15%	15-30%
Market	67.38	727	403	204	115
Model - Static	64.71	727	376	223	115
Model - Dynamic	42.78	551	308	197	85
	7Y				
	0-3%	3-7%	7-10%	10-15%	15-30%
Market	70.5	780	440	248	128.5
Model - Static	70.46	842	437	263	129.3
Model - Dynamic	56.49	781	431	274	128.5
	10Y				
	0-3%	3-7%	7-10%	10-15%	15-30%
Market	73.5	895.5	509	282	139.5
Model - Static	71.89	899.6	452	282	139.1
Model - Dynamic	60.76	929	511	319	144

Marginal Parameters - Static				
	θ	α	β_1	β_2
M	.4738	.4283	.0823	.3138
V	2.4119	-.1965	.1441	.5066

Other Parameters - Static	
x_0	ρ
2.5010	.9967

Initial Values - Dynamic		
x_0	M_0	V_0
2.6656	-0.9107	0.0942